

# Partial Radon Transform and Hamburger moment completion in $\mathbb{R}^2$

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**ABSTRACT.** The Radon transform is one of the most powerful tools for reconstructing data from a series of projections. Reconstruction of Radon transform with missing data can be closely related to reconstruction of a function from moment sequences with missing terms. A new range theorem is established for the Radon transform  $Rf$  on  $L^1$  based on the Hamburger moment problem in two variables, and the sparse moment problem is converted into the Radon transform with missing data and vice versa. A modified Radon transform for missing data is constructed and an inversion formula is established.

## 1. Introduction

**1.1. The Radon Transform in  $\mathbb{R}^2$ .** The Radon transform, which was introduced by Johann Radon in 1917, is the integral transform consisting of the integral of a function over either hyperplanes or straight lines. A key application for the Radon transform is tomography, which is a technique for reconstructing the interior structure of an object from a series of projections of this object and is based on deep pure mathematics and numerical analysis as well as science and engineering. In general, we will follow the notation in [9] and [13]. The Radon transform  $R$  in  $\mathbb{R}^2$  is defined by

$$(1.1) \quad Rf(w, p) = \int_{\langle x, \omega \rangle = p} f(x) dm(x),$$

where  $\omega = (\omega_1, \omega_2)$  is a unit vector,  $p \in \mathbb{R}$ , and  $dm$  is the arc-length measure on the line  $\langle x, \omega \rangle = p$  with the usual inner product  $\langle \cdot, \cdot \rangle$ . The Radon transform can be expressed as

$$(1.2) \quad Rf(w, p) = \int_{\mathbb{R}^2} f(x) \chi_{\{\langle x, \omega \rangle = p\}} dx,$$

where  $dx$  is the Lebesgue measure on  $\mathbb{R}^2$  and  $\chi$  is the indicator function.

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2010 *Mathematics Subject Classification.* Primary 44A12, 44A60, 47A57; Secondary 28A25, 44A17.

*Key words and phrases.* Radon transform, Hamburger Moment problems, and moment extension problems.

LEMMA 1.1. (see [9], Lemma 2.3). *For each  $f \in L^1(\mathbb{R}^2)$  the Radon transform  $Rf$  satisfies the following condition: For  $k \in \mathbb{N}_0$  the integral*

$$\int_{\mathbb{R}} Rf(\omega, p) p^k dp$$

*can be written as a  $k^{\text{th}}$  degree homogeneous polynomial in  $\omega_1, \omega_2$ .*

We denote the unit vector in direction  $\theta$  as  $\omega = \omega(\theta) := (\omega_1, \omega_2)$  with  $\omega_1 = \cos \theta$  and  $\omega_2 = \sin \theta$ . Thus, the Radon transform of  $f \in L^1(\mathbb{R}^2)$  can be expressed as a function of  $(\theta, p)$ :

$$(1.3) \quad Rf(\theta, p) = \int_{\langle x, \omega(\theta) \rangle = p} f(x) dm(x).$$

Note that since the pairs  $(\omega, p)$  and  $(-\omega, -p)$  give the same line,  $R$  satisfies the *evenness* condition:  $Rf(\theta, p) = Rf(\theta + \pi, -p)$ .

THEOREM 1.2. *The Radon transform  $R$  is a bounded linear operator from  $L^1(\mathbb{R}^2)$  to  $L^1([0, 2\pi] \times \mathbb{R})$  with norm  $\|R\| \leq 2\pi$ , i.e.,  $\|Rf\|_{L^1([0, 2\pi] \times \mathbb{R})} \leq 2\pi \|f\|_{L^1(\mathbb{R}^2)}$ .*

PROOF. See [13]. □

Along with the transform  $Rf$  we define the dual Radon transform of  $g \in L^1([0, 2\pi] \times \mathbb{R})$  as

$$(1.4) \quad R^*g(x) = \int_0^{2\pi} g(\theta, \langle x, \omega \rangle) d\theta,$$

which is the integral of  $g$  over all lines through  $x$ . Using  $F_1$  and  $F_2$  for the 1-D and 2-D Fourier transforms, recall that

$$\begin{aligned} F_1 f(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt, \\ F_1^{-1} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{ist} ds, \\ F_2 f(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) e^{-i\langle x, \xi \rangle} dx, \\ F_2^{-1} f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} f(\xi) e^{i\langle \xi, x \rangle} d\xi. \end{aligned}$$

THEOREM 1.3 (Projection-Slice Theorem). *Let  $f \in L^1(\mathbb{R}^2)$ . Then,*

$$F_2 f(s\omega) = \frac{1}{\sqrt{2\pi}} F_1(Rf(\theta, \cdot))(s).$$

This theorem shows that  $R$  is injective on  $L^1(\mathbb{R}^2)$ . The Fourier inversion formula combined with the Projection-Slice Theorem provides an inversion formula for  $f$  from  $Rf$ .

Denote the Riesz potential  $I^{-1}$ , for  $g \in L^1([0, 2\pi] \times \mathbb{R})$ , as the operator with Fourier multiplier  $|s|$  (see [11]):

$$(1.5) \quad I^{-1}g = F_1^{-1}(|s|(F_1 g(\theta, \cdot))(s)).$$

THEOREM 1.4 (Inversion formula for  $Rf$ ). *Let  $f \in C_c^\infty(\mathbb{R}^2)$ . Then*

$$f(x) = \frac{1}{4\pi} R^*(I^{-1}Rf)(x).$$

Note that this theorem is true on a larger domain than  $C_c^\infty(\mathbb{R}^2)$ . However,  $I^{-1}Rf$  can be a distribution rather than a function.

A function  $f$  is said to be in the *Schwartz space*  $\mathcal{S}(\mathbb{R}^2)$  if and only if  $f \in C^\infty(\mathbb{R}^2)$  and for each polynomial  $P$  and each integer  $m \geq 0$ ,

$$\sup_x ||x|^m P(\partial_1, \partial_2) f(x)| < \infty,$$

where  $|x|$  is the norm of  $x$ . A function  $g(\theta, p)$  is said to be in the Schwartz space  $\mathcal{S}([0, 2\pi] \times \mathbb{R})$  if  $g(\theta, p)$  can be extended to be smooth and  $2\pi$ -periodic in  $\theta$ , and  $g(\cdot, p) \in \mathcal{S}(\mathbb{R})$  uniformly in  $\theta$ .

One has the question of when a given function  $g$  is the Radon transform of a function  $f$ . In other words, for a given function  $g$ , does there exist  $f$  such that  $g = Rf$ ? The following theorem is the fundamental result on this question, which is called the *Schwartz theorem* or *Range Theorem* for the Radon transform.

**THEOREM 1.5.** *Let  $g \in \mathcal{S}([0, 2\pi] \times \mathbb{R})$  be even. Then, there exists  $f \in \mathcal{S}(\mathbb{R}^2)$  such that  $g = Rf$  if and only if*

$$(1.6) \quad \text{for each } k \in \mathbb{N}_0, \text{ the } k^{th} \text{ moment } \int_{-\infty}^{\infty} g(\theta, p) p^k dp$$

is a homogeneous polynomial of degree  $k$  in  $\omega_1$  and  $\omega_2$ .

**PROOF.** See [9], Theorem 2.4. □

**1.2. Hamburger Moment Problems in  $\mathbb{R}^2$ .** The *Hamburger moment sequence* of a positive measure  $\mu$  on  $\mathbb{R}^2$  is defined by

$$\gamma_{\alpha_1, \alpha_2} = \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} d\mu \quad \text{for all } \alpha_1, \alpha_2 \in \mathbb{N}_0,$$

assuming the integrals converge absolutely.

Let  $\mathcal{P}_2$  be the algebra of all polynomial functions on  $\mathbb{R}^2$  with complex coefficients. A linear map  $L : \mathcal{P}_2 \rightarrow \mathbb{C}$  is *positive semi-definite* if  $L(ff) \geq 0$  for all  $f \in \mathcal{P}_2$ . We assume  $L(1) > 0$ . A function  $L : \mathcal{P}_2 \rightarrow \mathbb{C}$  is a *moment* if there exists a positive measure  $\mu$  on  $\mathbb{R}^2$  such that

$$(1.7) \quad L(f) = \int_{\mathbb{R}^2} f d\mu \quad \text{for all } f \in \mathcal{P}_2.$$

In this case, the measure  $\mu$  is called a *representing measure* for  $L$ . A solution of a moment problem is called *determined* if the corresponding representing measure is unique. A 2-sequence  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is said to be *positive semi-definite (moment)* if there exists a positive semi-definite linear map (a moment function)  $L_\gamma : \mathcal{P}_2 \rightarrow \mathbb{C}$  such that  $L_\gamma(x_1^{\alpha_1} x_2^{\alpha_2}) = \gamma_{\alpha_1, \alpha_2}$  for  $\alpha_1, \alpha_2 \in \mathbb{N}_0$ , respectively.

It is trivial that every moment sequence is positive semi-definite. Hamburger established that a positive semi-definite 1-sequence is a moment sequence (see [1], Theorem 2.1.1.). However, in  $\mathbb{R}^2$ , there are positive semi-definite sequences which are not moment sequences (see [4] and [5]).

The Hamburger moment problem in  $\mathbb{R}^2$  is stated as:

Given a 2-sequence of real numbers  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$ , find a positive Borel measure  $\mu$  on  $\mathbb{R}^2$  such that

$$\gamma_{\alpha_1, \alpha_2} = \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} d\mu \quad \text{for all } \alpha_1, \alpha_2 \in \mathbb{N}_0.$$

A necessary and sufficient condition for existence of such a positive Borel measure  $\mu$  on  $\mathbb{R}^2$  is the following:

**THEOREM 1.6.** *A 2-sequence  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  ( $\gamma_0 > 0$ ) is a moment 2-sequence if and only if there exists a positive semi-definite 4-sequence*

$$\{\delta_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}\}_{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}_0}$$

*satisfying the following properties:*

- (1)  $\gamma_{\alpha_1, \alpha_2} = \delta_{\alpha_1, \alpha_2, 0, 0}$ ;
- (2)  $\delta_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} = \delta_{\alpha_1, \alpha_2, \alpha_3+1, \alpha_4} + \delta_{\alpha_1+2, \alpha_2, \alpha_3+1, \alpha_4}$ ;
- (3)  $\delta_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} = \delta_{\alpha_1, \alpha_2, \alpha_3, \alpha_4+1} + \delta_{\alpha_1, \alpha_2+2, \alpha_3, \alpha_4+1}$  for all  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}_0$ .

*In this case, the 2-sequence  $\gamma$  has a uniquely determined representing measure in  $\mathbb{R}^2$  if and only if the 4-sequence  $\delta$  is unique.*

PROOF. See [14] □

## 2. Moment completion problems

A *partial sequence* is a sequence in which some terms are specified, while the remaining terms are unspecified and may be treated as free real variables.

The *Hamburger moment completion* of a partial sequence is a specific choice of values for the unspecified terms resulting in the Hamburger moment sequence on  $\mathbb{R}$ .

The *pattern* of a partial sequence is the set of positions of the specified entries. Denote the pattern of a partial sequence by the set of ordered pairs of nonnegative integers

$$P = \{(i, j) : \gamma_{i, j} \text{ is specified}\}.$$

We say that a pattern  $P$  is *Hamburger moment completable* if every partial moment sequence with pattern  $P$  has a Hamburger moment completion.

For more information on the trigonometric multidimensional moment completion we refer to [3]. Note that Bakonyi and Naevdal deal with truncated Fourier coefficient problem and show the following Theorem.

A positive Borel measure  $\mu$  on  $\mathbb{T}^d$  is called a *positive extension* of  $\{c_k\}_{k \in \Gamma - \Gamma}$  if

$$c_k = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{-i\langle k, t \rangle} d\mu(t), \quad k \in \Gamma - \Gamma.$$

A finite subset  $\Gamma \subset \mathbb{Z}^d$  has the *extension property* if every sequence  $\{c_k\}_{k \in \Gamma - \Gamma}$  which is positive with respect to  $\Gamma$  admits a positive extension.

**THEOREM 2.1.** *A finite subset  $\Gamma \in \mathbb{Z}$  possesses the extension property if and only if it is an arithmetic progression.*

## 3. Partial Radon Transform and Hamburger moment completion

In practice it is often required to reconstruct images from partial Radon transform data. Function  $f$  to be reconstructed is the density of the object. So, we assume that  $f$  is nonnegative. It is clear that if  $f \geq 0$ , then so is  $Rf$ .

Denote the set of moment 2-sequences by  $\mathcal{M}$ . Let  $\mathcal{N}$  be the set of moment 2-sequences with a positive Borel measures  $\mu$  such that the linear subspace spanned by  $\{x_1^{\alpha_1} x_2^{\alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is dense in  $L^1(\mu)$ . It is trivial that  $\mathcal{N} \subset \mathcal{M}$ . Then, the following theorem holds.

**THEOREM 3.1.** *Let  $g \in L^1([0, 2\pi] \times \mathbb{R})$  be nonnegative and even. Then, there exists nonnegative  $f \in L^1(\mathbb{R}^2)$  such that  $g = Rf$  a.e. if and only if for each  $k \in \mathbb{N}_0$ ,*

$$(3.1) \quad \int_{-\infty}^{\infty} g(\theta, p) p^k dp = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2}$$

for some  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0} \in \mathcal{N}$ .

That is, it is a homogeneous polynomial in  $\omega_1$  and  $\omega_2$  of degree  $k$  whose coefficient is a certain multiple of  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0} \in \mathcal{N}$ .

**PROOF.** It is simple to prove the necessary part. Since  $\langle x, \omega \rangle^k = (x_1 \omega_1 + x_2 \omega_2)^k$ , by Lemma 1.1 and polynomial expansion, it follows that

$$\int_{\mathbb{R}} Rf(\omega, p) p^k dp = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \omega_1^{\alpha_1} \omega_2^{\alpha_2} \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} f(x_1, x_2) dx.$$

Since  $f \geq 0$ ,  $\int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} f(x_1, x_2) dx$  is a moment 2-sequence.

Conversely, assume that for each  $k \in \mathbb{N}_0$

$$(3.2) \quad \int_{-\infty}^{\infty} g(\theta, p) p^k dp = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2},$$

where  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0} \in \mathcal{N}$ .

Then, there is a positive Borel measure  $\mu$  such that

$$(3.3) \quad \gamma_{\alpha_1, \alpha_2} = \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} d\mu.$$

Let  $m$  be the Lebesgue measure on  $\mathbb{R}^2$ . By the Lebesgue-Radon-Nikodym Theorem, there exist uniquely nonnegative  $f \in L^1(m)$  and a Borel measure  $\nu$  such that  $\nu \perp m$  and

$$d\mu = f dm + d\nu.$$

Since  $\nu \perp m$ , there is a Borel set  $E \subset \mathbb{R}^2$  such that  $\nu(E^c) = 0$  and  $m(E) = 0$ . We claim that  $\int g d\nu = 0$  for all  $g \in L^1(\mu)$ . Suppose that  $\nu(E) > 0$ . Then, there exists  $h \in L^1(\mu)$  and  $\delta > 0$  such that

$$\int h d\nu \geq \delta.$$

Define  $\tilde{h} : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in E, \\ 0 & \text{if } x \in E^c. \end{cases}$$

Since  $m(E) = 0$ , for any polynomial  $P \in L^1(m)$ ,

$$(3.4) \quad \int_E P(x) dm = 0.$$

Then, it follows that

$$(3.5) \quad \int (\tilde{h} - P(x)) d\mu \geq \delta,$$

which is a contradiction, implying  $d\nu = 0$ .

Thus, for each  $\alpha_1, \alpha_2 \in \mathbb{N}_0$

$$(3.6) \quad \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} d\mu = \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} f(x_1, x_2) dm.$$

Then, by (3.3) and (3.6), one gets that

$$\begin{aligned} \int_{-\infty}^{\infty} g(\theta, p) p^k dp &= \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2} \\ &= \int_{\mathbb{R}^2} f(x) \langle x, \omega \rangle^k dx \\ &= \int_{-\infty}^{\infty} Rf(\theta, p) p^k dp \quad \text{for each } k \in \mathbb{N}_0. \end{aligned}$$

Thus, for each  $k \in \mathbb{N}_0$  and all  $\theta \in [0, 2\pi]$

$$\int_{-\infty}^{\infty} (g(\theta, p) - Rf(\theta, p)) p^k dp = 0.$$

Since  $Rf - g \in L^1([0, 2\pi] \times \mathbb{R})$ ,  $(Rf - g)(\theta, \cdot) = 0$  for all  $\theta \in [0, 2\pi]$ . Therefore,  $Rf = g$  a.e.  $\square$

The following is how to check whether  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is a moment 2-sequence: Suppose that for each  $k \in \mathbb{N}_0$ ,

$$(3.7) \quad \int_{-\infty}^{\infty} g(\theta, p) p^k dp = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2}$$

where  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is a 2-sequence.

Fix  $k \in \mathbb{N}_0$ . Then, (3.7) can be expressed as the following

$$(3.8) \quad b(\theta) = \sum_{j=0}^k C(k, j) (\cos^j \theta \sin^{k-j} \theta) \gamma_{j, k-j},$$

where

$$(3.9) \quad C(k, j) = \binom{k}{j} := \frac{k!}{j!(k-j)!} \quad \text{and}$$

$$(3.10) \quad b(\theta) := \int_{-\infty}^{\infty} g(\theta, p) p^k dp.$$

Let  $0 < \theta_k < \theta_{k-1} < \dots < \theta_0 < \frac{\pi}{2}$  be distinct angles. Then, system (3.8) can be written in matrix form as follows:

$$(3.11) \quad \mathbf{A} \mathbf{x} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{pmatrix} C(k, 0) \cos^0 \theta_0 \sin^k \theta_0 & C(k, 1) \cos^1 \theta_0 \sin^{k-1} \theta_0 & \dots & C(k, k) \cos^k \theta_0 \sin^0 \theta_0 \\ C(k, 0) \cos^0 \theta_1 \sin^k \theta_1 & C(k, 1) \cos^1 \theta_1 \sin^{k-1} \theta_1 & \dots & C(k, k) \cos^k \theta_1 \sin^0 \theta_1 \\ C(k, 0) \cos^0 \theta_2 \sin^k \theta_2 & C(k, 1) \cos^1 \theta_2 \sin^{k-1} \theta_2 & \dots & C(k, k) \cos^k \theta_2 \sin^0 \theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ C(k, 0) \cos^0 \theta_k \sin^k \theta_k & C(k, 1) \cos^1 \theta_k \sin^{k-1} \theta_k & \dots & C(k, k) \cos^k \theta_k \sin^0 \theta_k \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} \gamma_{0,k} \\ \gamma_{1,k-1} \\ \gamma_{2,k-2} \\ \vdots \\ \gamma_{k,0} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b(\theta_0) \\ b(\theta_1) \\ b(\theta_2) \\ \vdots \\ b(\theta_k) \end{pmatrix}.$$

The determinant of the matrix  $A$  can be expressed as:

$$\det(A) = \det(V) \prod_{i=1}^k (C(k, i) \sin^k \theta_i),$$

where  $V = [\cot^{j-1} \theta_i]_{1 \leq i, j \leq k+1}$  is a Vandermonde matrix. Using the Vandermonde determinant formula, it is easy to show  $\det(A)$  is positive, implying the system (3.11) has a unique solution  $\mathbf{x}$ . Note that the matrix  $A$  is positive definite since its leading principal minors are all positive.

The Equation (3.11) need to be considered for  $k = 0, 1, 2, \dots$ , in order to get all entries of  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$ . Note that countable number of angles are necessary and sufficient to find all entries of the 2-sequence  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$ . By Theorem 1.6, this can be checked if  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is a moment 2-sequence.

The following theorem states that for Theorem 3.1,  $k$  does not need to hold for all  $k \in \mathbb{N}_0$ . In other words,  $k$  can be in a subset of  $\mathbb{N}_0$ . For a pattern  $P$ , we define  $P_+ = \{i + j : (i, j) \in P\}$ .

**COROLLARY 3.2.** *Let  $g(\theta, p) \in L^1([0, 2\pi] \times \mathbb{R})$  be nonnegative, even and  $P$  be a Hamburger completable pattern. If for each  $k \in P_+$ ,*

$$(3.12) \quad \int_{-\infty}^{\infty} g(\theta, p) p^k dp = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1! \alpha_2!} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2}$$

*for some 2-sequence  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$ . Then, there exists nonnegative  $f \in L^1(\mathbb{R}^2)$  such that  $g = Rf$  a.e.*

Note that  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0}$  is not required to be a moment sequence. The completion Theorem guarantees there exists a moment completion.

#### 4. Modified Radon Transform

**DEFINITION 4.1.** If  $\varphi$  is a smooth function on  $\mathbb{R}$ , satisfying the following four requirements:

- (1) it is compactly supported,
- (2)  $\int_{\mathbb{R}} \varphi(t) dt = 1$ ,
- (3)  $\lim_{\varepsilon \rightarrow 0} \varphi_{\varepsilon}(t) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \varphi(\frac{t}{\varepsilon}) = \delta(t)$ ,
- (4)  $\varphi(t) \geq 0$ , for all  $t \in \mathbb{R}$ .

Then  $\varphi$  is called a **positive mollifier**.

Furthermore, if

- (5)  $\varphi(t) = h(|t|)$  for some infinitely differentiable function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , then it is called a **symmetric mollifier**.

**EXAMPLE 4.2.** Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then  $\varphi$  is a positive and symmetric mollifier.

Let  $\Omega = \{\varphi : \varphi \text{ is a positive and symmetric mollifier such that } F_1(\varphi)(s) > 0 \text{ for all } s \in \mathbb{R}\}$ . Clearly, the Gaussian function is in  $\Omega$ .

DEFINITION 4.3. Let  $\varphi \in \Omega$  and  $f \in L^1(\mathbb{R}^2)$ . The *modified Radon transform* in  $\mathbb{R}^2$  is defined by

$$(4.1) \quad \widehat{R}f(\theta, p) = \int_{\mathbb{R}^2} \chi(x; \theta, p) f(x) dx,$$

where

$$\chi(x; \theta, p) = (\delta * \varphi)(\langle x, \omega \rangle - p), \quad (\delta \text{ is a delta function}).$$

LEMMA 4.4.

$$(4.2) \quad \widehat{R}f(\theta, p) = \int_{\mathbb{R}} Rf(\theta, p + \tau) \varphi(\tau) d\tau$$

PROOF.

$$\begin{aligned} \widehat{R}f(\theta, p) &= \int_{\mathbb{R}^2} (\delta * \varphi)(\langle x, \omega \rangle - p) f(x) dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \delta(\langle x, \omega \rangle - p - \tau) f(x) dx \right) \varphi(\tau) d\tau \\ &= \int_{\mathbb{R}} \left( \int_{\langle x, \omega \rangle = p + \tau} f(x) dx \right) \varphi(\tau) d\tau. \end{aligned}$$

□

PROPOSITION 4.5.

$$\widehat{R}f(\theta, p) = (Rf(\theta, \cdot) * \varphi)(p).$$

PROOF. By Lemma 4.4

$$\begin{aligned} \widehat{R}f(\theta, p) &= \int_{\mathbb{R}} Rf(\theta, p + \tau) \varphi(\tau) d\tau \\ &= \int_{\mathbb{R}} Rf(\theta, p - \tau) \varphi(-\tau) d\tau. \end{aligned}$$

Since  $\varphi$  is a symmetric mollifier,

$$\widehat{R}f(\theta, p) = \int_{\mathbb{R}} Rf(\theta, p - \tau) \varphi(\tau) d\tau.$$

□

THEOREM 4.6. The modified Radon transform  $\widehat{R}$  is a bounded linear operator from  $L^1(\mathbb{R}^2)$  to  $L^1([0, 2\pi] \times \mathbb{R})$  with norm  $\|R\| \leq 2\pi$ , i.e.,  $\|\widehat{R}f\|_{L^1([0, 2\pi] \times \mathbb{R})} \leq 2\pi \|f\|_{L^1(\mathbb{R}^2)}$ .



PROOF. Using Proposition 4.5, one finds that

$$\begin{aligned}
\|\widehat{R}f\|_{L^1([0,2\pi]\times\mathbb{R})} &= \int_{\theta=0}^{2\pi} \int_{p=-\infty}^{\infty} \left| \widehat{R}f(\theta, p) \right| dp d\theta \\
&= \int_{\theta=0}^{2\pi} \int_{p=-\infty}^{\infty} \left| \int_{\tau=-\infty}^{\infty} Rf(\theta, p + \tau) \varphi(\tau) d\tau \right| dp d\theta \\
&\leq \int_{\theta=0}^{2\pi} \int_{p=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} \left| Rf(\theta, p + \tau) \right| |\varphi(\tau)| d\tau dp d\theta \\
&= \int_{\tau=-\infty}^{\infty} |\varphi(\tau)| \left( \int_{\theta=0}^{2\pi} \int_{p=-\infty}^{\infty} \left| Rf(\theta, p + \tau) \right| dp d\theta \right) d\tau.
\end{aligned}$$

By Theorem 1.2 and Definition 4.1, it follows that

$$\begin{aligned}
\|\widehat{R}f\|_{L^1([0,2\pi]\times\mathbb{R})} &\leq \int_{\tau=-\infty}^{\infty} |\varphi(\tau)| \left( 2\pi \|f\|_{L^1(\mathbb{R}^2)} \right) d\tau \\
&= 2\pi \|f\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

□

Denote the modified Riesz potential  $\widehat{I}^{-1}$ , for  $g \in L^1([0,2\pi]\times\mathbb{R})$ , as the operator with Fourier multiplier  $|s|$  and symmetric mollifier  $\varphi$ :

$$\widehat{I}^{-1}g = F_1^{-1} \left( |s| \left( \frac{F_1(g(\theta, \cdot))(s)}{F_1(\varphi)} \right) \right).$$

The proof of theorem 1.4 combined with the convolution theorem provides an inversion formula for  $f$  from  $\widehat{R}$ .

THEOREM 4.7 (Inversion formula for  $\widehat{R}f$ ). *Let  $f \in C_c^\infty(\mathbb{R}^2)$ . Then*

$$f(x) = \frac{1}{4\pi} R^* (\widehat{I}^{-1} \widehat{R}f)(x).$$

PROOF. By theorem 4.5 and the convolution theorem,

$$(4.3) \quad F_1(\widehat{R}f(\theta, \cdot))(s) = F_1(Rf(\theta, \cdot))(s) F_1(\varphi)(s).$$

Since  $F_1(\varphi)(s) \neq 0$ , by Theorem 1.3, it follows that

$$(4.4) \quad F_2 f(s\omega) = \frac{F_1(\widehat{R}f(\omega, \cdot))(s)}{\sqrt{2\pi} F_1(\varphi)(s)}$$

Applying the Fourier inversion formula and Proposition 4.5, one shows that

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} F_2 f(\xi) e^{i\langle x, \xi \rangle} d\xi \\
&= \frac{1}{2\pi} \frac{1}{2} \int_{\theta=0}^{2\pi} \int_{s=-\infty}^{\infty} F_2 f(s\omega) e^{i\langle x, s\omega \rangle} |s| ds d\theta \\
&= \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \int_{s=-\infty}^{\infty} \widehat{I}^{-1} \widehat{R}f(\theta, \langle x, \omega \rangle) d\theta \\
&= \frac{1}{4\pi} R^* (\widehat{I}^{-1} \widehat{R}f)(x).
\end{aligned}$$

□

Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  be a multi-index with  $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$  and  $\alpha! := \alpha_1! \alpha_2! \alpha_3!$ . The following is the Range theorem for  $\widehat{R}$ .

**THEOREM 4.8.** *Let  $g \in L^1([0, 2\pi] \times \mathbb{R})$  be nonnegative and even. Then, there exists nonnegative  $f \in L^1(\mathbb{R}^2)$  such that  $g = \widehat{R}f$  a.e. if and only if for each  $k \in \mathbb{N}_0$ ,*

$$(4.5) \quad \int_{-\infty}^{\infty} g(\theta, p) p^k dp = \sum_{|\alpha|=k} \frac{k!}{\alpha!} C_{\alpha_3} \gamma_{\alpha_1, \alpha_2} \omega_1^{\alpha_1} \omega_2^{\alpha_2}$$

where  $\{\gamma_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2 \in \mathbb{N}_0} \in \mathcal{N}$  and  $C_n = \int_{-\infty}^{\infty} \varphi(\tau) (-\tau)^n d\tau$  for each  $n \in \mathbb{N}_0$ .

**PROOF.** Suppose that  $g = \widehat{R}f$  a.e. for some  $f \in L^1(\mathbb{R}^2)$ . By Lemma 4.4,

$$\begin{aligned} \int_{p=-\infty}^{\infty} \widehat{R}f(\theta, p) p^k dp &= \int_{p=-\infty}^{\infty} \left( \int_{\tau=-\infty}^{\infty} Rf(\theta, p + \tau) \varphi(\tau) d\tau \right) p^k dp \\ &= \int_{p=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} \left( \int_{\langle x, \omega \rangle = p + \tau} f(x) dm(x) \right) \varphi(\tau) d\tau p^k dp \\ &= \int_{\tau=-\infty}^{\infty} \varphi(\tau) \left[ \int_{p=-\infty}^{\infty} p^k \left( \int_{\langle x, \omega \rangle = p + \tau} f(x) dm(x) \right) dp \right] d\tau \\ &= \int_{\tau=-\infty}^{\infty} \varphi(\tau) \left( \int_{\mathbb{R}^2} (\langle x, \omega \rangle - \tau)^k f(x) dx \right) d\tau. \end{aligned}$$

Use the following polynomial expansion

$$(\langle x, \omega \rangle - \tau)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (x_1 \omega_1)^{\alpha_1} (x_2 \omega_2)^{\alpha_2} (-\tau)^{\alpha_3},$$

the result follows. The proof of the converse part is similar to the proof of Theorem 3.1.  $\square$

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